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Physical Properties of Some McVittie Metrics

Henning Knutsen[1]

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A class of McVittie's new nonquadratic solutions is investigated in some detail with regard to its physical properties. It is found that decreasing pressure and density are not compatible with center regularity for these perfect fluid spheres. It is further seen that for "gaseous" spheres (i.e., the density ρ drops to zero at the outer boundary of the sphere together with the pressure p) oscillatory motions are *not* possible. For these "gaseous" models the pressure and the density are both positive inside the outer surface, and their respective gradients are negative. For the outer "gaseous" shells models are constructed where for a certain time interval the pressure is increasing for contracting models. Without any restriction with respect to time, for these shell models it found that the density is increasing for contracting models, and the adiabatic speed of sound is less than the speed of light. It is also found that the trace of the energy-momentum tensor is positive, the total mass is negative, and for collapsing shells the rate of change of circumference as measured by an observer riding on the shell is an increasing function of time. However, all these models have the strange geometric feature that the "physical radius" is a decreasing function of comoving radial coordinate.

1. INTRODUCTION

According to Einstein's general relativity, all gravitational fields are themselves sources of gravity. As a result, the equations governing gravity are nonlinear. These nonlinearities pose extremely difficult mathematical problems when one, for example, wants to construct exact models for the final, relativistic stage in the evolution of a star. A realistic discussion would require that we consider heat flow, radiation, neutrino energy transport, nuclear forces, magnetic fields, and even rotation. Such highly complicated systems may best be investigated with the help of numerical calculation. Important steps in this direction have already been taken (Matsuda and Sato, 1969; Shapiro and Teukolsky, 1980; Nakamura, 1981; Nakamura and Sato, 1981, 1982; Petrich *et al.*, 1985; Stark and Piran, 1985.

[1]Stavanger College of Engineering, Box 2527, Ullandhaug 4001, Stavanger, Norway.

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However, closed analytic solutions of Einstein's field equations should be interesting, since we cannot claim to have reliable knowledge of the unusual physical conditions under which matter exists during the last stages of gravitational collapse. To obtain simple, closed formulas for the gravitational potentials, density, pressure, etc., one has to consider situations with high symmetry.

A beautiful method for obtaining exact solutions of the field equations for a nonstatic perfect fluid sphere has been developed by McVittie (1967). However, McVittie's approach does not necessarily yield models which are physically meaningful. This has been the main motivation for my previous investigations of some of McVittie's elementary solutions (Knutsen and Stabell, 1979; Knutsen, 1982, 1983, 1984a). Recently McVittie (1984) has extended his solutions to functions that are not elementary, and I have also discussed (Knutsen, 1985c) a particular class of those nonelementary solutions with regard to its physical properties.

In this paper another class of nonelementary solutions is examined. For this class it is found that center regularity is not compatible with negative pressure and density gradients.

Since the heavenly bodies generally are gaseous spheres where the density vanishes at the surface of the sphere, I have previously (Knutsen, 1984b, 1985a,b) constructed several models of that kind. Models for such gaseous spheres are also considered in this paper, and it is found that such gaseous spheres may exist with negative pressure and density gradients. For the outer layers of these gaseous spheres I have also constructed models where even more physical conditions are fulfilled:

1. The pressure and the density are increasing functions of time for contracting spheres.
2. The adiabatic speed of sound is less than the speed of light.
3. The energy condition $\rho > 3p$ holds.
4. The total mass is negative.
5. For collapsing mass shells, the rate of change of circumference as measured by an observer riding on the spherical shell is an increasing function of time.

It also turns out that the "physical radius" is a decreasing function of comoving radial coordinate. This strange geometric feature is of course connected with the fact that these models are not regular at the center.

2. THE McV METRIC

The metrics McVittie considers for the nonstatic perfect fluid sphere are of the form

$$ds^2 = y^2\, dt^2 - S^2(t) e^{\eta}[dr^2 + f^2(r)(d\theta^2 + \sin^2\theta\, d\varphi^2)] \tag{1}$$

where y and η are functions of a variable z defined by

$$e^z = Q/S \tag{2}$$

where Q is a function of the comoving coordinate r alone.

For a perfect fluid the following equation is now obtained from $T_4^1 = 0$ (T_μ^ν here denotes the energy-momentum tensor):

$$y = 1 - \tfrac{1}{2}\eta_z \tag{3}$$

The subscript here denotes differentiation with respect to z.

Pressure isotropy now yields three differential equations (McVittie, 1967):

$$\frac{Q''}{Q} - \frac{Q'f'}{Qf} = a\left(\frac{Q'}{Q}\right)^2 \tag{4}$$

$$\frac{f''}{f} - \frac{f'^2}{f^2} + \frac{1}{f^2} = b\left(\frac{Q'}{Q}\right)^2 \tag{5}$$

$$y_{zz} + (a-3+y)y_z + [a+b-2-(a-3)y-y^2]y = 0 \tag{6}$$

where a prime denotes differentiation with respect to r, and a and b are constants.

Using Einstein's equations "in reverse," McVittie also found that the density ρ and the pressure p for the McV metric (1) are given by

$$8\pi G\rho = 3\left(\frac{\dot{S}}{S}\right)^2 + \frac{e^{-\eta}}{S^2}\left\{3\frac{1-f'^2}{f^2} - 6(1-y)\frac{f'Q'}{fQ} - [2b-2y_z+(1-y)(2a-1-y)]\left(\frac{Q'}{Q}\right)^2\right\} \tag{7}$$

$$8\pi Gp = \frac{1}{y}\left\{-2\frac{\ddot{S}}{S} - (3y-2)\left(\frac{\dot{S}}{S}\right)^2 - \frac{e^{-\eta}}{S^2}\left[y\frac{1-f'^2}{f^2} + 2(y^2-y-y_z)\frac{f'Q'}{fQ} + (1-y)(y^2-y-2y_z)\left(\frac{Q'}{Q}\right)^2\right]\right\} \tag{8}$$

where a dot denotes differentiation with respect to time.

3. THE NONELEMENTARY SOLUTION

The equations (4) and (5) for f and Q were dealt with by McVittie (1967), but there only certain elementary solutions of equation (6) were found through the discovery that y_z could be a quadratic function of y. However, McVittie (1984) later solved equation (6) under certain conditions in terms of elliptic functions. In this paper I will discuss his solution:

$$y^2 = 9\beta^2 \frac{1-\sin\beta z}{(1+\sin\beta z)(2-\sin\beta z)^2} \tag{9}$$

$$e^{\eta} = e^{\varepsilon+2z}\frac{(1+\sin\beta z)^2}{(2-\sin\beta z)^2} \tag{10}$$

where

$$a = 3 \tag{11}$$

$$b = \beta^2 - 1 \tag{12}$$

and ε is an arbitrary integration constant.

Inserting these results into equation (6), we find in fact that the only valid solution is

$$y = -3\beta\left(\frac{1-X}{1+X}\right)^{1/2}\frac{1}{2-X} \tag{13}$$

with

$$y_z = 3\beta^2 \frac{(X-\frac{1}{2})^2+\frac{3}{4}}{(X-2)^2(1+X)} \tag{14}$$

where

$$X = \sin\beta z \tag{15}$$

4. IRREGULARITY AT THE ORIGIN

The pressure gradient is most easily found from the equation that represents conservation of linear momentum, i.e., $T_1{}^{\mu}{}_{;\mu} = 0$ (a semicolon here denotes covariant differentiation), which yields

$$p' = -\frac{y_z Q'}{yQ}(\rho + p) \tag{16}$$

The general expression for the density gradient has been given in a previous paper (Knutsen, 1983). For the present model one finds

$$8\pi G\rho' = \frac{10e^{-\eta}}{S^2}\frac{Q'}{Q}(y_z + y^2 - \beta^2)\left(\frac{f'Q'}{fQ} + \frac{Q'^2}{Q^2}\right) \tag{17}$$

We observe that $y_z > 0$ and

$$y_z + y^2 - \beta^2 = -\beta^2 \frac{X^3 - 6X^2 + 12X - 12}{(X-2)^2(1+X)} < 0 \tag{18}$$

for $-1 < X < 1$.

Demanding that the pressure and the density be positive and their respective gradients be negative, one has from (16) and (17)

$$Q'/y > 0 \tag{19}$$

and

$$(Q'/Q)(f'Q'/fQ + Q'^2/Q^2) < 0 \tag{20}$$

However, to have regularity at the origin, it must be the case that the "physical radius"

$$R = e^{\eta/2} Sf \tag{21}$$

vanishes at the center, i.e., $f(0) = 0$ (Misner and Sharp, 1964).

Hence, we have

$$f'/f > 0 \tag{22}$$

If we now have $Q' > 0$, this immediately yields

$$(Q'/Q)(f'Q'/fQ + Q'^2/Q^2) > 0 \tag{23}$$

in contradiction to inequality (20).

Hence, let us consider $Q' < 0$. Inequalities (19) and (20) now yield

$$(Q'/Q + f'/f) < 0 \tag{24}$$

and

$$y < 0 \tag{25}$$

Differentiating equation (21), we find, however,

$$R' = R[(Q'/Q + f'/f) - yQ'/Q] \tag{26}$$

hence, we have $R' < 0$, which is not compatible with vanishing of the positive function R at the origin. The following statement has thus been proved: *Regularity at the center is not compatible with negative pressure and density gradients.*

5. ON THE POSSIBILITY OF PULSATIONS

We have previously (Knutsen and Stabell (1979) given a method for obtaining the scale function S from the junction condition that the internal solution should be fitted to an external vacuum Schwarzschild solution, i.e., the pressure p drops to zero at the boundary of the fluid sphere. Applying that method, we obtain after some straightforward calculations

$$\dot{S}^2 = e^{-\varepsilon} S^2 \left(\frac{Y-2}{Y+1}\right)^2 \times \left[A\frac{1}{Y+1} - B\left(\frac{1-Y}{1+Y}\right)^{1/2}\frac{1}{Y-2} - C\frac{Y-1}{(Y-2)^2(Y+1)} + D\frac{Y-2}{Y+1}\right] \quad (27)$$

where

$$A = 3\left(-\frac{1-f'^2}{f^2} + 2\frac{f'Q'}{fQ} + \frac{Q'^2}{Q^2}\right)_b \quad (28)$$

$$B = 6\beta\left(\frac{f'Q'}{fQ} + \frac{Q'^2}{Q^2}\right)_b \quad (29)$$

$$C = 9\beta^2\left(\frac{Q'^2}{Q^2}\right)_b \quad (30)$$

D is an arbitrary integration constant and

$$Y = X_b \quad (31)$$

Henceforth, boundary values will be denoted by the subscript b. I have also followed McVittie and Stabell (1968) and without loss of generality put $Q_b = 1$.

The global motion of the fluid sphere may be studied using equation (27). In addition, this equation also yields a consistency relation which must be fulfilled, i.e., $\dot{S}^2 \geq 0$.

Applying equations (13), (16), and (17), we find that we should demand

$$\beta\left(\frac{f'Q'}{fQ} + \frac{Q'^2}{Q^2}\right) > 0 \quad (32)$$

to arrive at a physically reasonable model.

Hence, we must demand

$$B > 0 \quad (33)$$

Necessary oscillatory conditions are then that the function within square brackets on the right-hand side of equation (27), call it $E(Y)$, has at least two positive roots, and that $E > 0$ between these two roots. It is thus immediately seen that pulsations are not possible if $A > 0$ and $D < 0$.

However, these conditions are fulfilled if we choose A, B, C, and D such that

$$E(-1)<0, \qquad E(0)>0, \qquad E(1)<0 \tag{34}$$

i.e., we choose A, B, C, and D in the following way:

$$D<\tfrac{1}{2}A+\tfrac{1}{4}B+\tfrac{1}{8}C \tag{35}$$

$$D>A \tag{36}$$

and

$$D>\tfrac{1}{3}A+\tfrac{2}{27}C \tag{37}$$

Hence, we first choose the positive quantities B and C. Then we choose

$$-\tfrac{2}{3}B-\tfrac{11}{36}C<A<\tfrac{1}{2}B+\tfrac{1}{4}C \tag{38}$$

and at last we choose

$$\mathrm{Max}(A,\tfrac{1}{3}+\tfrac{2}{27}C)<D<\tfrac{1}{2}A+\tfrac{1}{4}B+\tfrac{1}{8}C \tag{39}$$

It should be emphasized, however, that these conditions are not sufficient to have a physically acceptable oscillatory model.

6. "GASEOUS" SPHERES

From now on we demand that the sphere be "gaseous," i.e., we demand

$$\rho_b \equiv 0 \tag{40}$$

From equation (7) the following equation is now obtained for the scale function S:

$$\dot{S}^2=e^{-\varepsilon}S^2\left(\frac{Y-2}{Y+1}\right)^2\left[\frac{1}{3}A-B\left(\frac{1-Y}{1+Y}\right)^{1/2}\frac{1}{Y-2}+\frac{1}{27}\frac{2Y^2-14Y+11}{(Y-2)^2}\right] \tag{41}$$

Comparing equations (27) and (41), it is seen that for "gaseous" spheres the arbitrary integration constant D take the following value:

$$D=\frac{1}{3}A+\frac{2}{27}C=\left[-\frac{1-f'^2}{f^2}+2\frac{f'Q'}{fQ}+\left(1+\frac{2}{3}\beta^2\right)\frac{Q'^2}{Q^2}\right]_b \tag{42}$$

That the matching condition $p_b \equiv 0$ is fulfilled without more ado for gaseous spheres is of course no surprise, since the equation that represents conservation of energy, i.e., $T_4{}^{\mu}{}_{;\mu}=0$, yields

$$\dot{\rho}=-3y\dot{S}/S(\rho+p) \tag{43}$$

7. NONEXISTENCE OF PULSATING "GASEOUS" SPHERES

I now prove that oscillatory motions are forbidden for these "gaseous" spheres. Differentiating the function within square brackets on the right-hand side of equation (41) gives

$$\frac{d}{dY}\left[\frac{1}{3}A-B\left(\frac{1-Y}{1+Y}\right)^{1/2}\frac{1}{Y-2}+\frac{1}{27}C\frac{2Y^2-14Y+11}{(Y-2)^2}\right]$$
$$=-B\frac{(Y-\frac{1}{2})^2+\frac{3}{4}}{(1-Y^2)^{1/2}(2-Y)^2(1+Y)}-\frac{2}{9}C\frac{1+Y}{(2-Y)^3} \tag{44}$$

which is seen to be negative when we remember inequality (33).

Hence, equation (41) can have at most one root

$$Y_0\in\langle -1, 1\rangle$$

Strictly speaking, this could be enough to have an oscillatory model, since $Y=-\sin(\beta \ln S)$ could take the value Y_0 for two different values of S. However, then there would exist a moment when $Y=-1$ or $Y=1$. The situation $Y=-1$ is forbidden because the metric would then be singular. For $Y=1$ it is seen from equation (44) that equation (41) would then have no root at all.

Hence, *pulsations are not possible.*

8. INTEGRATION OF ISOTROPY EQUATIONS (4) AND (5)

Equation (4) is immediately integrated by quadrature to give

$$f=A_1(Q'/Q^3) \tag{45}$$

where A_1 is an arbitrary integration constant.

Following McVittie (1967), let us introduce a new radial coordinate q by

$$q=-\tfrac{1}{2}A_1Q^{-2} \tag{46}$$

which yields

$$dq/dr=f \tag{47}$$

Equation (5) may now be written

$$f_{qq}+\frac{1}{f^3}=\frac{\beta^2-1}{4}\frac{f}{q^2} \tag{48}$$

Following McVittie (1967) once more, equation (48) is integrated by the double substitution

$$q=e^w, \qquad f=e^{w/2}v(w) \tag{49}$$

I will just discuss models for which q is restricted to take positive values.

Hence, it follows from equations (21), (45), and (46) that only models with $Q' < 0$ will be taken into consideration.

From equation (49) we find

$$f_{qq} = \frac{1}{e^{3w/2}}\left(\frac{d^2v}{dw^2} - \frac{1}{4}v\right) \tag{50}$$

When this is inserted into equation (48) and the integration is performed, we get

$$\text{case 1:} \qquad f^2 = 2\frac{q}{\beta}(1 + \delta q^{\beta}) \tag{51}$$

or

$$\text{case 2:} \qquad f^2 = 2\frac{q}{\beta}(1 + \delta q^{-\beta}) \tag{52}$$

Here δ is an arbitrary integration constant, and suitable choice has been made of another integration constant. We also have

$$Q = (q_b/q)^{1/2} \tag{53}$$

We further find

$$-\frac{1-f'^2}{f^2} + 2\frac{f'Q'}{fQ} + \frac{Q'^2}{Q^2} = \begin{cases} (\beta/2q)(\delta q^{\beta} - 1) & \text{(case 1)} \\ (\beta/2q)(\delta q^{-\beta} - 1) & \text{(case 2)} \end{cases} \tag{54}$$

and

$$\frac{f'Q'}{fQ} + \frac{Q'^2}{Q^2} = \begin{cases} -\frac{1}{2}\delta q^{\beta-1} & \text{(case 1)} \\ \frac{1}{2}\delta q^{-\beta-1} & \text{(case 2)} \end{cases} \tag{55}$$

Case 1: Let us define

$$F = \beta/q_b, \qquad G = \delta q_b^{-\beta} \tag{56}$$

and obtain

$$A = \tfrac{3}{2}F(G-1), \qquad B = 3FG, \qquad C = \tfrac{9}{2}F(G+1) \tag{57}$$

For this case B and C are positive quantities, and the oscillatory condition (38) is also fulfilled if we choose

$$F > 0, \qquad G > 1/59 \tag{58}$$

or

$$F < 0, \qquad G < -7/3 \tag{59}$$

Case 2: The analysis is very much the same as in the previous case, but we put $G = \delta q_b^{\beta}$ and have $B = -3FG$. We find that we should take

$$F > 0, \qquad -1 < G < -1/13 \tag{60}$$

Observe, however, that the conditions (58) and (60) are not compatible with having a negative density gradient.

9. DENSITY, PRESSURE, AND PRESSURE GRADIENT

For this model the density ρ is given by

$$8\pi G e^{\varepsilon}\rho = H_b - H \tag{61}$$

where H is defined in the following way:

$$\begin{aligned} H = {} & \frac{3}{Q^2}\left(-\frac{1-f'^2}{f^2}+2\frac{f'Q'}{fQ}+\frac{Q'^2}{Q^2}\right)\left(\frac{X-2}{X+1}\right)^2 \\ & +\frac{18}{Q^2}\beta\left(\frac{f'Q'}{fQ}+\frac{Q'^2}{Q^2}\right)\frac{2-X}{X+1}\left(\frac{1-X}{1+X}\right)^{1/2} \\ & +\frac{\beta^2}{Q^2}\frac{Q'^2}{Q^2}\frac{2X^2-14X+11}{(X+1)^2} \end{aligned} \tag{62}$$

Further, one finds (after some calculation) that the pressure p is given by

$$\begin{aligned} & -8\pi G e^{\varepsilon}\left(\frac{1-X}{1+X}\right)^{1/2}\frac{1}{2-X}p \\ & = \left\{\left(-\frac{1-f'^2}{f^2}+2\frac{f'Q'}{fQ}+\frac{Q'^2}{Q^2}\right)_b\right. \\ & \quad \times\left[2\left(\frac{1-Y}{1+Y}\right)^{1/2}\frac{Y-2}{(Y+1)^2}-3\left(\frac{1-X}{1+X}\right)^{1/2}\frac{(Y-2)^2}{(X-2)(Y+1)^2}\right] \\ & \quad \left.+\frac{1}{Q^2}\left(-\frac{1-f'^2}{f^2}+\frac{f'Q'}{fQ}+\frac{Q'^2}{Q^2}\right)\left(\frac{1-X}{1+X}\right)^{1/2}\frac{X-2}{(X+1)^2}\right\} \\ & \quad +2\beta\left\{\left(\frac{f'Q'}{fQ}+\frac{Q'^2}{Q^2}\right)_b\left[-\frac{Y^2-7Y+7}{(Y+1)^3}\right.\right. \\ & \quad \left.+9\left(\frac{(1-X)(1-Y)}{(1+X)(1+Y)}\right)^{1/2}\frac{Y-2}{(X-2)(Y+1)^2}\right] \\ & \quad \left.+\frac{1}{Q^2}\left(\frac{f'Q'}{fQ}+\frac{Q'^2}{Q^2}\right)\frac{X^2+2X-2}{(X+1)^3}\right\} \\ & \quad +\beta^2\left\{\left(\frac{Q'}{Q}\right)_b^2\left[2\frac{Y-2}{(Y+1)^3}(1-Y^2)^{1/2}\right.\right. \\ & \quad \left.+\frac{2Y^2-14Y+11}{(Y+1)^2(2-X)(X+1)}(1-X^2)^{1/2}\right] \\ & \quad \left.+\frac{3}{Q^2}\left(\frac{Q'}{Q}\right)^2\frac{2X-1}{(2-X)(X+1)^3}\right\} \end{aligned} \tag{63}$$

From this equation it is easily seen that $p_b = 0$.

Using equation (16) we find that the pressure gradient may be written

$$8\pi G e^{\varepsilon} p' = -2\frac{y_z Q'}{yQ}\left(\frac{1+X}{1-X}\right)^{1/2}(X-2)(I-I_b) \tag{64}$$

where I is given by

$$I=\frac{1}{Q^2}\left[\left(-\frac{1-f'^2}{f^2}+2\frac{f'Q'}{fQ}+\frac{Q'^2}{Q^2}\right)\left(\frac{1-X}{1+X}\right)^{1/2}\frac{2-X}{(X+1)^2}\right.$$
$$\left.+\beta\left(\frac{f'Q'}{fQ}+\frac{Q'^2}{Q^2}\right)\frac{X^2-7X+7}{(X+1)^3}+\beta^2\frac{Q'^2}{Q^2}\frac{2-X}{(X+1)^2}\right] \tag{65}$$

To have a model with negative pressure gradient one would like to prove $(I-I_b)<0$.

Further, it is immediately seen from equations (17) and (55) that a proper choice of the constant δ yields *models with a negative density gradient.*

Hence, for our gaseous models this choice yields models where *the density is positive inside the boundary of the sphere.*

10. "GASEOUS" MODELS WITH POSITIVE PRESSURE

As usual, the main difficulty is to show that the pressure is positive throughout the sphere. From the complicated equation (63) the reader may get the feeling that this is a rather involved task. From equation (16) and the matching condition $p_b \equiv 0$, it is seen that for the present "gaseous" models this is equivalent to proving that the pressure gradient is negative throughout the sphere. I now show that the analysis is miraculously simple.

Differentiating equation (65), we in fact obtain

case 1: $$f^2=2\frac{q}{\beta}(1+\delta q^{\beta})$$

$$I'=-\frac{5}{4}\frac{\beta^2\delta f}{q_b}q^{\beta-1}\frac{(Y-2)^2}{(Y+1)^4}(1-Y^2)^{1/2}>0 \tag{66}$$

and

case 2: $$f^2=2\frac{q}{\beta}(1+\delta q^{-\beta})$$

$$I'=\frac{5}{4}\frac{\beta^2\delta f}{q_b}q^{-\beta-1}\frac{(Y-2)^2}{(Y+1)^4}(1-Y^2)^{1/2}>0 \tag{67}$$

Hence, *the pressure gradient is negative and the pressure is positive throughout the "gaseous" sphere.*

11. MODELS FOR SPHERICAL GAS SHELLS

Let us now investigate the outer layers of these time-dependent "gaseous" spheres.

Differentiating equation (16) once more, we obtain

$$p_b'' = -\left(\frac{y_z Q'}{yQ}\right)_b \rho_b' \tag{68}$$

Since we have $\rho_b' < 0$, the pressure is thus a convex function for the layers close to the boundary. We also have $p_b' = 0$ and this yields that the energy condition $\rho > 3p$ is fulfilled for the boundary layers.

The formula for the mass function for the McV metric has been given previously (Knutsen, 1985b). The total mass M is given by

$$M = -\frac{20}{3}\pi G \rho_b' R_b^3 \left(\frac{f'}{f} + \frac{Q'}{Q}\right)_b^{-1} \tag{69}$$

The total mass is thus seen to be negative.

11.1. Rate of Change of Circumference

Differentiating equation (21), we find

$$\dot{R} = e^{\eta/2} f y \dot{S} \tag{70}$$

McVittie (1967) and Nariai (1968) take v_M, where

$$v_M^2 = e^{\eta} f^2 \dot{S}^2 \tag{71}$$

to be the matter velocity. However, this "velocity" is in fact the rate of change of $1/2\pi$ time the circumference as measured by an observer riding in a shell of matter. From equation (41) we now obtain

$$\begin{aligned} \frac{d}{dt}(v_M^2)_b &= \beta(1-Y^2)^{1/2}\frac{\dot{S}}{S} f_b^2 \left[6\beta\left(\frac{f'Q'}{fQ} + \frac{Q'^2}{Q^2}\right)_b \frac{(Y-\frac{1}{2})^2 + \frac{3}{4}}{(1-Y^2)^{1/2}(Y-2)^2(1+Y)} \right. \\ &\left. + 2\beta^2\left(\frac{Q'}{Q}\right)_b^2 \frac{Y+1}{(2-Y)^3}\right] \end{aligned} \tag{72}$$

Recalling equations (13) and (74), we can now conclude that for expanding spheres the rate of change of circumference is decreasing, and for collapsing spheres it is increasing, with respect to time.

11.2. Time Development of Density and Pressure

From equations (43) and (70) we can immediately conclude that the density is increasing for contracting spheres and is decreasing for expanding spheres.

To show that there exists a certain time interval such that $\dot{p}(r \approx r_b) < 0$ for expanding models, it is enough to show that we have $\dot{p}_b'' < 0$. Differentiating equation (16) twice, we find

$$\dot{p}_b'' = \left[-\frac{\partial}{\partial t}\left(\frac{y_z}{y}\right) + 3y_z \frac{\dot{S}}{S} \right]_b \left(\frac{Q'}{Q}\right)_b \rho_b'$$
$$= \left(\frac{Q'}{Q}\right)_b \beta^2 \frac{\dot{S}}{S} \left[\frac{-8Y^3 + 21Y^2 - 24Y + 10}{(2-Y)^2(1-Y^2)} \right]_b \rho_b' \tag{73}$$

From equation (73) it is easily seen that $\dot{p}_b'' < 0$ for negative values of Y, i.e., $Y \in \langle -1, 0]$. Recalling equations (2) and (15), we see that S should be restricted to take values in the interval $[\ln Q_{\text{center}}, \pi/2\beta\rangle$.

11.3. Speed of Sound versus Speed of Light

The adiabatic speed of sound v_s for a nonstatic sphere is given by (Knutsen, 1984b)

$$v_s^2 = \dot{p}/\dot{\rho} \tag{74}$$

To see that this speed is less than the speed of light, i.e.,

$$(v_s^2)_b < 1 \tag{75}$$

it is enough to prove that the following relation holds for expanding models:

$$\dot{p}_b < \dot{\rho}_b \tag{76}$$

We now obtain

$$(\dot{p}' - \dot{\rho}')_b = 3y_b \frac{\dot{S}}{S} \rho_b' \tag{77}$$

Recalling equation (70), we can thus conclude that the adiabatic speed of sound is less than the speed of light.

12. CONCLUSION

A particular model of McVittie's nonelementary solutions of an isotropy equation has been investigated. It is found that the general expressions for pressure, density, etc., are quite complicated. However, this class of solutions certainly contains models for gaseous spheres which are quite acceptable from a physical point of view, i.e., the pressure and the density are positive, their respective gradients are negative, etc. The main objection is that these models have to be irregular at the origin, and this center irregularity yields a strange geometric feature: the "physical radius" is a decreasing function of comoving radial coordinate. But as long as an accepted theory for quantum gravity does not exist, one should not discard these models as physically meaningless. There may exist extreme astrophysical conditions where these models would be quite interesting, for example, during the last stages of gravitational collapse.

Finally, I emphasize the result that the "gaseous" models cannot describe pulsating spheres.

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